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Dynamic Optimization with a Non-Convex Technology: The Case of a Linear Objective Function

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The paper studies the problem of optimal intertemporal allocation in an aggregative model with a non-convex technology set and a discounted sum of consumptions as the objective function. The study demonstrates the existence of a threshold initial stock such that the long-run behaviour of optimal programmes depends critically on whether the initial stock is above or below the threshold. This is in contrast with the standard turnpike theory of convex models in which the long-run behaviour of optimal programmes is independent of the initial stock.

1. INTRODUCTION

In this note we reconsider the dynamic optimization exercise of Clark (1971) which, mathematically speaking, deals with maximization of a linear function on a particular type of non-convex set in the linear space of all bounded real sequences (l_∞). While Clark was concerned with the relation between profit-maximization and extinction of a competitive fishery, the exercise and the analytical results he discussed are clearly of wider interest, and directly applicable to the problem of optimal intertemporal allocation in an aggregative model which allows for an initial phase of increasing returns in the technology and uses a discounted sum of consumptions as the objective function. Following a precise statement of the optimization problem in Section 2, the main results characterizing optimal programmes are systematically presented in Section 3. We prove a conjecture by Clark on the existence of a "threshold" initial stock such that the long run behaviour of optimal programmes depends critically on whether the initial stock is above or below the threshold. This is in contrast with the standard "turnpike theory" of convex models in which the long run behaviour of optimal programmes is *independent* of the initial stock. Our primary motivation is to complete Clark's analysis and to emphasize the difference between dynamic optimization in convex and non-convex models.

2. THE OPTIMIZATION PROBLEM

We now state the infinite horizon optimization problem in a precise manner. A feasible programme from $x > 0$[1] is a sequence $\langle x \rangle = \langle x_t \rangle$ of non-negative reals satisfying $x_0 = x$,

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$0 \leqq x_{t+1} \leqq f(x_t)$ for $t \geqq 0$ where the function $f: R^+ \to R^+$ is assumed to have the following properties:

Assumption 1. $f(0)=0$; f is twice continuously differentiable with $f'(x)>0$ on R^+, $f'(\infty)<1<f'(0)<\infty$,

Assumption 2. There is k_1 such that (i) $f''(k_1)=0$, (ii) $f''(x)>0$ for $0 \leqq x<k_1$, (iii) $f''(x)<0$ for $x>k_1$.

The sequence of consumptions $\langle c \rangle = \langle c_t \rangle$ associated with a feasible programme, or simply, a feasible $\langle x \rangle = \langle x_t \rangle$, is defined as $c_t \equiv f(x_{t-1})-x_t$ for $t \geqq 1$. The value of $\langle x \rangle$, denoted by $V(\langle x \rangle)$, is defined as $V(\langle x \rangle) \equiv \sum_{t=1}^{\infty} \delta^{t-1} c_t$, where δ is the given discount factor, satisfying $0<\delta<1$. A feasible $\langle x^* \rangle = \langle x_t^* \rangle$ from $x>0$ is optimal if $V(\langle x^* \rangle) \geqq V(\langle x \rangle)$ for *all* feasible $\langle x \rangle$ from x. The problem is to derive qualitative properties of optimal programmes.

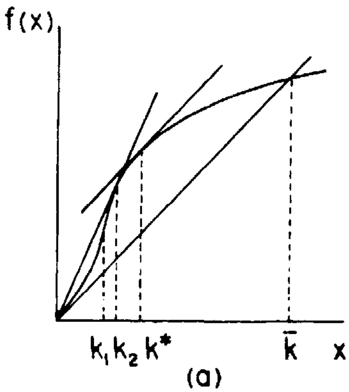
We refer to the problem just posed as a "non-classical" or "non-convex" problem to distinguish it from the corresponding "classical" or "convex" dynamic optimization problem which is obtained by replacing Assumption 2 with

Assumption 2'. $f''(x)<0$ for $x>0$.

and keeping all other features unchanged. In the classical problem, the set of all feasible programmes is a convex subset in the linear space of all real sequences (in fact, of l_∞). In our non-classical problem, this is not the case, and new arguments are needed to solve the problem.

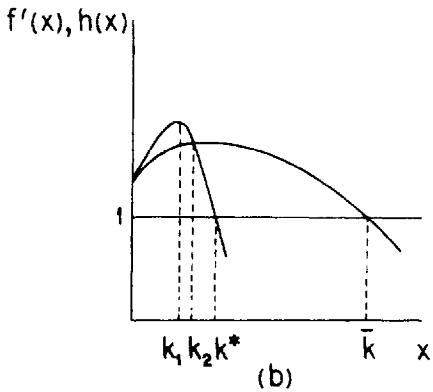
Define the average product function $h: R^+ \to R^+$ as $h(x) \equiv f(x)/x$ for $x>0$, $h(0)=\lim_{x \downarrow 0}[f(x)/x]=f'(0)$.

Under Assumptions 1 and 2, there exist uniquely determined real numbers, k^*, k_2, $\bar{k}$, satisfying (i) $0<k_1<k_2<k^*<\bar{k}<\infty$; (ii) $f'(k^*)=1$; (iii) $f(\bar{k})=\bar{k}$; (iv) $f'(k_2)=h(k_2)$. Furthermore, for $0 \leqq x<k^*$, $f'(x)>1$, and for $x>k^*$, $f'(x)<1$; for $0<x<\bar{k}$, $x<f(x)<\bar{k}$, and for $x>\bar{k}$, $\bar{k}<f(x)<x$; for $0<x<k_2$, $f'(x)>h(x)$, and for $x>k_2$, $f'(x)<h(x)$. Also, note that for $0 \leqq x<k_2$, $h(x)$ is increasing, and for $x>k_2$, $h(x)$ is decreasing; for $0 \leqq x<k_1$, $f'(x)$ is increasing, and for $x>k_1$, $f'(x)$ is decreasing. The functions f, f' and h, together with the numbers k_1, k_2, k^* and $\bar{k}$ may be represented diagrammatically as follows, in Figures 1(a) and 1(b).



Production Function

FIGURE 1(a)



Marginal & Average Product Functions

FIGURE 1(b)

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A feasible $\langle x\rangle$ from $\boldsymbol{x}>0$ is called stationary if $x_t = x_{t+1}$ for $t \geqq 0$. It is called an optimal stationary programme (OSP), if it is stationary, and optimal. The *pure accumulation programme* $\langle \bar{x}\rangle$ from $\boldsymbol{x}$ is defined as $\bar{x}_{t+1} = f(\bar{x}_t)$ for $t \geqq 0$. The extinction programme $\langle \tilde{x}\rangle$ from $\boldsymbol{x}$ is defined as $\tilde{x}_0 = \boldsymbol{x}$, $\tilde{x}_t = 0$ for $t \geqq 1$.

We note that for any feasible $\langle x\rangle$ from $\boldsymbol{x}>0$, we have $x_t \leqq \hat{k}$ and $c_{t+1} \leqq \hat{k}$, where $\hat{k} = \max(\boldsymbol{x}, \bar{k})$. Using Majumdar (1976, Theorem 1), one asserts that there *exists* an optimal programme from *every* $\boldsymbol{x}>0$. Without loss of generality we shall restrict $\boldsymbol{x}$ to belong to $(0, \bar{k})$. Then any feasible $\langle x\rangle = \langle x_t\rangle$ from any $\boldsymbol{x} \in (0, \bar{k})$ satisfies $(x_t, c_{t+1}) < (\bar{k}, \bar{k})$ for all $t \geqq 0$.

Interpretation

The non-classical exercise can be interpreted as a standard Ramsey-type aggregative optimal growth problem in which the gross output function f has an initial phase of increasing returns. The sequence $\langle x_t\rangle$ is the sequence of inputs or capital stocks. Variants of the corresponding convex problem dealing with maximization of a discounted sum of consumptions has been studied by many authors (e.g. Malinvaud (1965), Srinivasan (1964)). The paper by Majumdar and Mitra (1980*a*)[2] deals with the non-convex gross output function f (satisfying Assumptions 1 and 2) and examines—among other problems—the question of maximizing a discounted sum of utilities derived from consumption. However, the main assumptions on u (that u is strictly concave and satisfies $\lim_{c\downarrow 0} u'(c) = \infty$) clearly exclude the linear case $u(c) = c$ studied in this note and are critically used in the proofs there. The new techniques needed to handle the linear case actually lead to sharper results and a more complete characterization of the turnpike behaviour than the one obtained for the strictly concave utility function.

An alternative interpretation of the model is that of a competitive fishery (see Clark (1976) Chapter 7). According to this interpretation, x_t is the stock of fish in period t; the function f is the biological reproduction relationship or the "stock recruitment" function. The sequence $\langle c\rangle = \langle c_t\rangle$ is the sequence of "harvests". Let the profit per unit of harvesting, denoted by $q>0$ and the rate of interest $\gamma>0$ remain constant over time. Consider a firm which has an objective of maximizing the discounted sum of profits from harvesting. A feasible $\langle x^*\rangle = \langle x_t^*\rangle$ of stocks from $\boldsymbol{x}>0$ is optimal if

$$\sum_{t=1}^{\infty} \left[\frac{q}{(1+\gamma)^{t-1}}\right] c_t^* \geqq \sum_{t=1}^{\infty} \left[\frac{q}{(1+\gamma)^{t-1}}\right] c_t$$

for every feasible programme $\langle x\rangle$ from $\boldsymbol{x}$. This is exactly the problem posed above if we set $\delta = 1/(1+\gamma)$. Models of this type have been used to discuss the possible conflict between profit-maximization and conservation of natural resources (see, e.g. Clark (1971), Spence (1973) and Dasgupta-Heal (1979, Chapter V)).

3. CHARACTERIZATION OF OPTIMAL PROGRAMMES

The main results characterizing optimal programmes will now be stated and briefly discussed. Proofs of new results are sketched in the last section.[3]

In the qualitative analysis that follows, the roots of the equation $\delta f'(x) = 1$ play an important role. The equation might not have a non-negative real root at all; if it has a unique non-negative real root, we denote it by Z; if it has two non-negative real roots, the smaller one is denoted by a, the larger by Z.

The qualitative behaviour of optimal programmes depends on the value of δ, the discount factor. For convenience of exposition, we distinguish three cases. The first two have already been analyzed (and, interpreted in the context of a profit-maximizing fishery) by Clark (1971).

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3a. *Strong discounting*: $\delta f'(k_2) \leqq 1$

In this case, δ is "sufficiently small" ($\delta \leqq 1/f'(k_2)$)— in the fishery example, $1+\gamma \geqq f(x)/x$ for all $x > 0$.

Proposition 1. *The extinction programme, $\langle \tilde{x} \rangle$ is optimal from any $\boldsymbol{x} > 0$ and is the unique optimal programme if $\delta f'(k_2) < 1$.*

Remark 1. If $\delta f'(k_2) = 1$, an optimal programme need not be unique. To confirm this, let $\boldsymbol{x} = k_2$, and consider the feasible $\langle x^* \rangle$ from $\boldsymbol{x}$ defined as $x_t^* = \boldsymbol{x}$ for $t \geqq 0$. One verifies that

$$V(\langle x^* \rangle) = [f(k_2) - k_2]/(1-\delta) = f(\boldsymbol{x}) = V(\langle \tilde{x} \rangle).$$

Hence, both $\langle x \rangle$ and $\langle \tilde{x} \rangle$ are optimal from k_2. In fact, the feasible programmes $\langle x^n \rangle$ ($n = 1, 2, \ldots$) defined as $x_t^n = \boldsymbol{x}$ for $t = 0, \ldots, n$ and $x_t^n = 0$ for $t > n$ are all optimal.[4]

Remark 2. Consider the corresponding "classical" model of dynamic optimization satisfying Assumptions 1 and 2′. It is still true that if $\delta f'(0) \leqq 1$ in the classical model, the extinction programme is the unique optimal programme from every $\boldsymbol{x}$.

3b. *Mild discounting*: $\delta f'(0) \geqq 1$

In this case, δ is "sufficiently close to one" ($\delta \geqq 1/f'(0)$) and $Z > k_2$; if a exists, $a = 0$. When $\boldsymbol{x} < Z$, let M be the smallest integer such that $\bar{x}_M \geqq Z$ (i.e. M is the first period such that the pure accumulation programme $\langle \bar{x}_t \rangle$ from $\boldsymbol{x} > 0$ attains Z).

Proposition 2. *If $\boldsymbol{x} \geqq Z$, the feasible $\langle x^* \rangle$ defined as $x_0^* = \boldsymbol{x}$, $x_t^* = Z$ for $t \geqq 1$ is the unique optimal programme.*

Proposition 3. *If $\boldsymbol{x} < Z$, the feasible $\langle x^* \rangle$ defined as $x_0^* = \boldsymbol{x}$, $x_t^* = \bar{x}_t$ for $t = 1, \ldots, M-1$, $x_t^* = Z$ for $t \geqq M$, is the unique optimal programme.*

Remark 3. In the corresponding "classical" model, if $\delta f'(0) > 1$, there is a unique positive solution Z' to the equation $\delta f'(x) = 1$. Proposition 2 and Proposition 3 continue to hold with Z replaced by Z' (also in the definition of M).

3c. *Two turnpikes and the critical point of departure* ($\delta f'(0) < 1 < \delta f'(k_2)$)

In case (a), the programme $x_t = 0$ for all $t \geqq 0$, and in case (b), the optimal stationary programme $x_t = Z$ for all $t \geqq 0$, serve as the "turnpike" attained by optimal programmes. *Both* the classical and the non-classical models share the feature that the long run behaviour of optimal programmes is independent of initial stocks. We come now to the most interesting—and difficult—part of our analysis in which the contrast between convex and non-convex models is quite remarkable. In this case, $0 < a < k_1 < k_2 < Z < k^*$. The arguments leading to Proposition 3 also apply to the following:

Proposition 4. *If $\boldsymbol{x} \geqq Z$, the feasible $\langle x^* \rangle$ defined as $x_0^* = \boldsymbol{x}$, $x_t^* = Z$ for $t \geqq 1$, is optimal.*

We are left with the situation in which the initial stock $\boldsymbol{x}$ is actually less than Z. A precise characterization of the long-run behaviour of optimal programmes requires careful reasoning.

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Lemma 1. *There is a unique OSP $\langle x^* \rangle$ defined as $x_t^* = Z$ for $t \geqq 0$.*

Define $A = \{x \in (0, Z)$: the extinction programme from x is an optimal programme from $x\}$. We shall show that A is a non-degenerate interval.

Lemma 2. *$x \in A$ implies $(0, x) \in A$.*

A feasible $\langle x \rangle$ from $x < Z$ is called a regeneration programme if there is some integer $N \geqq 1$ such that $x_t > x_{t-1}$ for $1 \leqq t \leqq N$, and $x_t = Z$ for $t \geqq N$. It should be stressed that a regeneration programme may allow for positive consumption in *all* periods, and need not be a "pure accumulation" programme in the initial periods. For an example of a regeneration programme that is also optimal and allows for positive consumption, the reader is referred to Clark (1971, p. 259).

Lemma 3. *If x does not belong to A, and $\langle x \rangle$ is an optimal programme from x, then $\langle x \rangle$ is a regeneration programme.*

From Lemma 3 we are led to

Lemma 4. *A is non-empty.*

Finally, note that there is a unique $d > 0$ such that $a < d < k_2$ and $f(d)/d = 1/\delta$.

Lemma 5. *If $x \in (d, z)$, the extinction $\langle \tilde{x} \rangle$ is not optimal from x.*

By Lemma 2 and Lemma 4, A is a non-degenerate interval. Define $\hat{k} = \sup A$. By Lemma 4, $\hat{k} > 0$, and $\hat{k} \leqq d$. Thus, the qualitative behaviour of optimal programmes can be described as follows. There is a "threshold" or "critical stock" level $\hat{k} > 0$ such that if $0 < x < \hat{k}$, the extinction $\langle \tilde{x} \rangle$ from x is an optimal programme; if $\hat{k} < x < Z$, then any optimal programme from x is a regeneration programme. The long-run behaviour of optimal programmes depends crucially on the initial stock; figuratively speaking, there are two turnpikes, and the point of departure determines which one is reached in a finite number of periods.

Remark 4. In the literature on competitive fishery, $\hat{k}$ is called the "minimum safe standard of conservation" (see Clark (1971)). It has been argued that a conservation policy which prohibits economic exploitation of a fishery till the stock of the fishery exceeds $\hat{k}$, will ensure that the fishery will not become extinct, even under pure "economic exploitation".

Remark 5. It can be checked that if $0 < f(x) < \hat{k}$ then the extinction programme from x is the unique optimal programme from x.

Remark 6. If $d < x < Z$, then it can be shown that a "pure regeneration programme" is the unique optimal programme from x. That is, the optimal programme $\langle x \rangle$ from x is a regeneration programme from x, and $x_t = \bar{x}_t$ for $0 \leqq t < N$, $x_t = Z$ for $t \geqq N$. This means that there is no consumption during the period of regeneration to the OSP, Z.

Remark 7. It can be shown that if $x = \hat{k}$, then the extinction programme, and a regeneration programme are optimal from x. This information may be used to compute $\hat{k}$ in actual situations where f is numerically specified, and the discount factor δ is known.

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4. PROOFS

Since the proofs of Proposition 1 through Proposition 4 can be constructed from Clark (1971), we sketch the proofs of the remaining assertions in Section 3c, i.e. in what follows we are assuming $\delta f'(0) < 1 < \delta f'(k_2)$.

Note, first, that for any feasible $\langle x \rangle$ from $\mathbf{x} > 0$, $V(\langle x \rangle) \equiv \sum_{t=1}^{\infty} \delta^{t-1} c_t = J_1 + f(\mathbf{x})$ where $J_1 \equiv \sum_{t=1}^{\infty} \delta^{t-1} \phi(x_t)$ and $\phi(x) \equiv \delta f(x) - x$. It is easy to show that

Proposition 5. *$\phi(Z) \geqq \phi(x)$ for $x \geqq 0$, and $\phi(Z) > \phi(x)$ for $x \neq Z$.*

Define the function $W: R^+ \to R$ as

$$W(x) \equiv \max \{J_1: 0 \leqq x_1 \leqq f(x); \tag{1}$$

$0 \leqq x_t \leqq f(x_{t-1})$ for $t \geqq 2\}$.

One can verify that (since f is increasing).

Proposition 6. *$W(x) \geqq 0$ for $x \geqq 0$, W is increasing.*

Using (1) as the principle of optimality one can conclude:

Proposition 7. *If $\langle x^* \rangle$ is optimal from $\mathbf{x} > 0$, then given any $T > 0$, the sequence $\langle x' \rangle$ defined as $x'_t = x^*_{t+T}$ for $t \geqq 0$ is optimal from x^*_T.*

Proof of Lemma 1. In view of Proposition 4, uniqueness remains to be established. If $\langle x \rangle$ is a stationary programme from any $\mathbf{x} > 0$, for it to be an OSP, $f(\mathbf{x}) - \mathbf{x} > 0$ and $U(x) \equiv [f(\mathbf{x}) - x] + \delta[f(x) - \mathbf{x}]$ must be maximized at $x = \mathbf{x}$ among all x satisfying $f(\mathbf{x}) \geqq x$, $f(x) \geqq \mathbf{x}$. Since the maximum is at an interior point $\delta f'(\mathbf{x}) = 1$. But $f''(a) > 0$ implies that $\mathbf{x} = Z$. ||

Proof of Lemma 2. Follows easily from Proposition 6.

Proof of Lemma 3. Several steps are needed. We first prove a sequence of Lemmas and then complete the proof.

Lemma 6. *If $\langle x^* \rangle$ is optimal from $\mathbf{x} < Z$, $x^*_t \leqq Z$, for all $t \geqq 0$.*

Proof. Suppose Lemma 6 is false, and let s be the first period with $x^*_t > Z$, for an optimal $\langle x^* \rangle$ from some $\mathbf{x} < Z$. Define a feasible $\langle x' \rangle$ from $\mathbf{x}$ by $x'_t = x^*_t$, $t = 0, \ldots, s-1$ and $x'_t = Z$ for $t \geqq s$. Now use Proposition 5 to verify that $V(\langle x' \rangle) > V(\langle x^* \rangle)$, a contradiction to the optimality of $\langle x^* \rangle$. ||

Lemma 7. *If $\langle x^* \rangle$ is optimal from $\mathbf{x} < Z$ and $0 < x^*_{t+1} \leqq x^*_t$ for some $t = s > 0$, then $x^*_{s+1} = Z$.*

Proof. By Proposition 7, the sequences $\langle x' \rangle$ defined as $x'_t = x^*_{t+s}$ for $t \geqq 0$ and $\langle x'' \rangle$ defined as $x''_t = x^*_{t+s+1}$ for $t \geqq 0$ are optimal from x^*_s and x^*_{s+1} respectively.

One computes that

$$V(\langle x' \rangle) = [f(x^*_s) - x^*_{s+1}] + \delta V(x''). \tag{2}$$

Define $\langle \hat{x} \rangle$ as $\hat{x}_0 = x^*_s$, $\hat{x}_t = x''_t$ for all $t \geqq 1$. Observe that $\hat{c}_1 = f(x^*_s) - x''_1 = f(x^*_s) - x^*_{s+2} = f(x^*_s) - f(x^*_{s+1}) + f(x^*_{s+1}) - x^*_{s+2} = f(x^*_s) - f(x^*_{s+1}) + c^*_{s+2} \geqq 0$ (using the hypothesis $x^*_{s+1} \leqq x^*_s$ and the monotonicity of f). For $t \geqq 2$, $\hat{c}_t = c''_t = c^*_{s+t+1} \geqq 0$. Thus,

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$\langle \hat{x} \rangle$ is feasible from x_s^*. By optimality of $\langle x' \rangle$ from x_s^*,

$$V(\langle x' \rangle) \geqq V(\langle \hat{x} \rangle)$$

or,

$$\begin{aligned} V(\langle x' \rangle) &\geqq \hat{c}_1 + \sum_{t=2}^{\infty} \delta^{t-1} c_t'' \\ &= f(x_s^*) - f(x_{s+1}^*) + c_{s+2}^* + \sum_{t=2}^{\infty} \delta^{t-1} c_t'' \\ &= f(x_s^*) - f(x_{s+1}^*) + V(\langle x'' \rangle). \end{aligned} \tag{3}$$

From (2) and (3) we have

$$V(\langle x'' \rangle) \leqq [f(x_{s+1}^*) - x_{s+1}^*]/(1-\delta). \tag{4}$$

Finally, the sequence $\langle \hat{\hat{x}} \rangle$ defined as $\hat{\hat{x}}_t = x_{s+1}^*$ for $t \geqq 0$ is feasible from x_{s+1}^*. Moreover, $V(\langle \hat{\hat{x}} \rangle) = [f(x_{s+1}^*) - x_{s+1}^*]/(1-\delta)$. From (4) and optimality of $\langle x'' \rangle$, we conclude that $\langle \hat{\hat{x}} \rangle$ must be optimal from x_{s+1}^*. Since $\langle \hat{\hat{x}} \rangle$ is stationary, Lemma 1 implies that $x_{s+1}^* = Z$. ‖

Lemma 8. *Suppose that* $\boldsymbol{x} < Z$ *and* $\boldsymbol{x} \notin A$. *If* $\langle x^* \rangle$ *is optimal from* $\boldsymbol{x} > 0$ *and* $\boldsymbol{x} \leqq x_{t-1}^* < Z$ *for some* $t = s \geqq 1$, $x_{s-1}^* < x_s^*$.

Proof. Suppose that Lemma 8 is false and $x_s^* \leqq x_{s-1}^*$. Then, $x_s^* < Z$. First, note that $x_s^* > 0$. Otherwise, if $x_s^* = 0$, so is x_{s+t}^* for $t \geqq 1$. From Proposition 7, $x_{s-1}^* \in A$, and $\boldsymbol{x} \leqq x_{s-1}^*$ implies $\boldsymbol{x} \in A$, a contradiction. Thus, we have $0 < x_s^* \leqq x_{s-1}^* < Z$, and Lemma 7 is contradicted. ‖

We now complete the proof of Lemma 3. By Lemma 6, either (a) $x_t < Z$ for all t, or (b) $x_t = Z$ for some t. If (a) holds, Lemma 8 leads to $x_{t+1} > x_t$ for all t, which, in turn, implies that x_t converges to some $\hat{x}$. Clearly, $\boldsymbol{x} \leqq \hat{x} \leqq Z$.

So c_t converges to $[f(\hat{x}) - \hat{x}] > 0$. Hence, we can find T large enough, so that for $t \geqq T$, $c_t > 0$. Since $\langle x \rangle$ is optimal, so for each $t \geqq T$, the expression

$$U(x) = [f(x_{t-1}) - x] + \delta[f(x) - x_{t+1}]$$

is maximized at $x = x_t$, among all x satisfying $f(x_{t-1}) \geqq x$, $f(x) \geqq x_{t+1}$. Since the maximum is at an interior point, $U'(x_t) = 0$, $U''(x_t) \leqq 0$ for $t \geqq T$. This means $\delta f'(x_t) = 1$ for $t \geqq T$, and $\delta f''(x_t) \leqq 0$ for $t \geqq T$. Hence $x_t = Z$ for $t \geqq T$, a contradiction. So (a) cannot occur. Hence (b) must occur.

Let M be the smallest integer for which $x_M = Z$. Then $M \geqq 1$, and $x_t < Z$ for $t = 0, \ldots, M-1$. Again, Lemma 8 leads to $x_{t+1} > x_t$ for $t = 0, \ldots, M-1$.

By Proposition 7, the sequence $\langle x' \rangle$ defined as $x_t' = x_{t+M}$ for $t \geqq 0$, is optimal from x_M. Since $x_M = Z$, $x_{t+M} = Z$ for $t \geqq 0$.

To summarize, there is an integer $M \geqq 1$, such that $x_t > x_{t-1}$ for $1 \leqq t \leqq M$, and $x_t = Z$ for $t \geqq M$. Hence $\langle x \rangle$ is a regeneration programme. ‖

Proof of Lemma 4. Denote $[1/\delta h(a)]$ by m. Since $\delta h(a) = \delta f(a)/a < 1$, $m > 1$, $\delta < \delta m$, and $\delta m = [1/h(a)] < 1$. Denote $[m-1]/m$ by $\hat{m} > 0$.

Since $\delta m < 1$, we can choose a positive integer M such that $(\delta m)^M < a$. Choose $T > M$, such that

$$(\delta m)^{T+M} \geqq [2\delta^T \bar{k}/\hat{m}], \quad (\delta m > \delta \text{ ensures this can be done}) \tag{5}$$

$$\sum_{t=1}^{T} \delta^{t-1} \geqq \frac{1}{2(1-\delta)} \quad \left(\text{as } \sum_{t=1}^{\infty} \delta^{t-1} = \frac{1}{1-\delta}, \text{ this can be done}\right). \tag{6}$$

Finally, choose $\boldsymbol{x} = [\delta m]^{T+M}$.

Consider the pure accumulation programme $\langle \bar{x} \rangle$ from $\boldsymbol{x}$. Then we claim that $\bar{x}_t < a$ for $t = 0, 1, \ldots, T$. This is clearly true for $t = 0$, since $\boldsymbol{x} = [\delta m]^{T+M} < [\delta m]^M < a$. Suppose

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the claim is true for $t=0,\ldots,n$, where $n<T$. Then, we have $\bar{x}_{n+1}=f(\bar{x}_n)=[f(\bar{x}_n)/\bar{x}_n]\bar{x}_n \leqq [f(a)/a]\bar{x}_n=[\delta h(a)][\bar{x}_n/\delta]=\bar{x}_n/(\delta m)$. Iterating on this relationship, $\bar{x}_{n+1}\leqq \bar{x}_0/(\delta m)^{n+1}=(\delta m)^{T+M}/(\delta m)^{n+1}\leqq(\delta m)^{T+M}/(\delta m)^T=(\delta m)^M<a$. This completes the induction step, and establishes our claim that $\bar{x}_t<a$ for $t=0,1,\ldots,T$.

Suppose $\boldsymbol{x}\notin A$. By Lemma 3, any optimal $\langle x\rangle$ from $\boldsymbol{x}$ must be a regeneration programme. Then for *all* $t\geqq 0$, $\boldsymbol{x}<x_t<\bar{x}_t$. In particular, for $t=0,\ldots,T$, $\boldsymbol{x}<x_t<\bar{x}_t<a$. For $2\leqq t\leqq T+1$, we have

$$\delta^{t-1}c_t=\delta^{t-1}f(x_{t-1})-\delta^{t-1}x_t$$
$$=[\delta^{t-2}x_{t-1}-\delta^{t-1}x_t]-\delta^{t-2}x_{t-1}\left[1-\frac{\delta f(x_{t-1})}{x_{t-1}}\right].$$

Now, for $2\leqq t\leqq T+1$, we have

$$\delta f(x_{t-1})/x_{t-1}\leqq \delta f(a)/a<1,$$

so

$$\delta^{t-2}x_{t-1}\left[1-\frac{\delta f(x_{t-1})}{x_{t-1}}\right]\geqq \delta^{t-2}x_{t-1}\left[1-\frac{\delta f(a)}{a}\right]\geqq \delta^{t-2}\boldsymbol{x}\hat{m}.$$

Hence, for $2\leqq t\leqq T+1$, we have

$$\delta^{t-1}c_t\leqq[\delta^{t-2}x_{t-1}-\delta^{t-1}x_t]-\delta^{t-2}\boldsymbol{x}\hat{m},$$

and

$$\textstyle\sum_{t=2}^{T+1}\delta^{t-1}c_t\leqq x_1-\delta^T x_{T+1}-\boldsymbol{x}\hat{m}\sum_{t=2}^{T+1}\delta^{t-2}.$$

So,

$$\textstyle\sum_{t=1}^{T+1}\delta^{t-1}c_t\leqq f(\boldsymbol{x})-\delta^T x_{T+1}-\boldsymbol{x}\hat{m}\sum_{t=2}^{T+1}\delta^{t-2}$$
$$\textstyle\leqq f(\boldsymbol{x})-\boldsymbol{x}\hat{m}\sum_{t=1}^{T}\delta^{t-1}\leqq f(\boldsymbol{x})-[(\boldsymbol{x}\hat{m})/2(1-\delta)]\quad\text{[by (6)]}.$$

Also,

$$\textstyle\sum_{t=T+2}^{\infty}\delta^{t-1}c_t\leqq\bar{k}\sum_{t=T+2}^{\infty}\delta^{t-1}=\bar{k}\delta^{T+1}\sum_{t=1}^{\infty}\delta^{t-1}=\bar{k}\delta^{T+1}/(1-\delta).$$

Hence

$$\textstyle\sum_{t=1}^{\infty}\delta^{t-1}c_t\leqq f(\boldsymbol{x})-\dfrac{\boldsymbol{x}\hat{m}}{2(1-\delta)}+\dfrac{\bar{k}\delta^{T+1}}{(1-\delta)}.$$

Now,

$$\boldsymbol{x}=(\delta m)^{T+M}\geqq\frac{2\delta^T\bar{k}}{\hat{m}}\quad\text{[by (5)]},$$

so,

$$\frac{\boldsymbol{x}\hat{m}}{2(1-\delta)}\geqq\left[\frac{2\delta^T\bar{k}}{\hat{m}}\right]\left[\frac{\hat{m}}{2(1-\delta)}\right]=\frac{\delta^T\bar{k}}{(1-\delta)}>\frac{\delta^{T+1}\bar{k}}{(1-\delta)}.$$

So, $V(\langle x\rangle)\equiv\sum_{t=1}^{\infty}\delta^{t-1}c_t<f(\boldsymbol{x})=V(\langle\tilde{x}\rangle)$. This contradiction establishes that $\boldsymbol{x}\in A$. ||

Proof of Lemma 6. For any $\boldsymbol{x}$ satisfying $d<\boldsymbol{x}<Z$, consider the stationary feasible $\langle x\rangle$ from $\boldsymbol{x}$ defined by $x_t=\boldsymbol{x}$ for $t\geqq 0$. Check that $V(\langle x\rangle)=[f(\boldsymbol{x})-\boldsymbol{x}]/(1-\delta)$. Now, $f(\boldsymbol{x})/\boldsymbol{x}>f(d)/d=1/\delta$ implies $V(\langle x\rangle)>f(\boldsymbol{x})=V(\langle\tilde{x}\rangle)$. Hence the extinction $\langle\tilde{x}\rangle$ from $\boldsymbol{x}$ cannot be optimal. ||

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NOTES

1. We denote the set of non-negative reals by R^+, the set of reals by R.
2. For comprehensive discussions of non-convexity, dynamic optimization and renewable resources with references to the literature, the reader may turn to Majumdar and Mitra (1980*a*), Clark (1971), Dasgupta and Heal (1979).
3. Proofs are spelled out in detail in the self-contained working paper by Majumdar and Mitra (1980*b*).
4. This example illustrates the fact that the statement of Theorem 5 in Clark (1971, p. 256) is not quite correct. Clark claims that any optimal programme is either the extinction programme, or a sustained yield programme (i.e. has the property that for some $\bar{k} > k' > 0$, $x_t = k'$ for $t \geqq T$ where T is a positive integer). Note that the feasible programmes $\langle x^n \rangle$ for $n \geqq 2$, are not extinction programmes (as defined by us, or by Clark), and they are not sustained yield programmes either. However, each $\langle x^n \rangle$ is optimal. Each $\langle x^n \rangle$ has the property that it leads to the extinction of the fishery after a finite number of periods, $n \geqq 2$, but not to "immediate" extinction.

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